Observable subgroups of algebraic monoids

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Abstract

A closed subgroup H of the affine, algebraic group G is called *observable* if G/H is a quasi-affine algebraic variety. In this paper we define the notion of an observable subgroup of the affine, algebraic *monoid* M. We prove that a subgroup H of G is observable in M if and only if H is closed in M and there are "enough" H-semiinvariant functions in $\Bbbk[M]$. We show also that a closed, normal subgroup H of G (the unit group of M) is observable in M if and only if it is closed in M. In such a case there exists a *determinant* $\chi : M \to \Bbbk$ such that $H \subseteq \ker(\chi)$. As an application, we show that in this case the *affinized quotient* $M/_{\mathrm{aff}}\, H$ of M by H is an affine algebraic monoid scheme with unit group G/H.

1. Introduction

A closed subgroup H of the affine algebraic group G is called an *observable subgroup* if the homogeneous space G/H is a quasi-affine variety. Such subgroups have been researched extensively, notably by F. Grosshans, see [5] for a survey on this topic, and Theorem 2.12 below for other useful characterizations of observable subgroups. In [10] the authors presented the notion of an *observable action* of G on the affine variety X, together to its basic properties. In this paper we develop further the notion of observable actions. In particular, we investigate the situation where M is an affine *algebraic monoid* with unit group G, and H is a closed subgroup of G, such that the action of

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H on M by left multiplication is observable. In this case, we say that H is *observable in* M.

We describe now the organization of this paper.

In Section 2 we provide the basic definitions and properties of observable actions and affinized quotients. In Section 3 we give several characterizations of observable subgroups. In Theorem 3.3 we deduce a number of useful consequences from the assumption that H is observable in M. In particular it follows that H is an observable subgroup of G, and that it is closed in M. In Theorem 3.4 we characterize the observable subgroups of M in terms of semiinvariants. In Theorem 3.5 we show that H is an observable subgroup of M if and only if H is the isotropy group of some vector $v \in V$ in some rational representation $\rho : M \to \mathrm{End}(V)$ of M. In the final section (Section 4) we use the results of the previous section to study the affinized quotient of an affine algebraic monoid by a closed normal subgroup. In Theorem 4.4 we show that if H is a closed, normal subgroup of G, closed in M, then H is observable in M. Whether this is true for nonnormal closed subgroups of G is an open question. See Remark 4.7. As an application, we show that the *affinized quotient* of an affine algebraic monoid M by a normal subgroup H, closed in M, is an algebraic monoid, with unit group G/H.

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2. Preliminaries

Let $\Bbbk$ be an algebraically closed field. We work with affine algebraic varieties X over $\Bbbk$. An algebraic group is assumed to be a smooth, affine, group scheme of finite type over $\Bbbk$. If X is an affine variety over $\Bbbk$ we denote by $\Bbbk[X]$ the ring of regular functions on X. If $I \subset \Bbbk[X]$ is an ideal, we denote by $\mathcal{V}(I) = \{x \in X : f(x) = 0 \ \forall f \in I\}$. If $Y \subset X$ is a subset, we denote by $\mathcal{I}(Z) = \{f \in \Bbbk[X] : f(y) = 0 \ \forall y \in Y\}$. Morphisms $\varphi : X \to Y$ between affine varieties correspond to morphisms of algebras $\Bbbk[Y] \to \Bbbk[X]$, by $\varphi \mapsto \varphi^*$, $\varphi^*(f) = f \circ \varphi$. If X is irreducible we denote by $\Bbbk(X)$ the field of rational functions on X. If A is any integral domain we denote by $[A]$ its quotient field. Thus if X is an irreducible affine variety, then $\Bbbk(X) = \big[\Bbbk[X]\big]$.

Let G be an affine algebraic group and let X be an algebraic variety. A *(regular) action* of G on X is a morphism $\varphi : G \times X \to X$, denoted by $\varphi(g, x) = g \cdot x$, such that $(ab) \cdot x = a \cdot (b \cdot x)$ and $1 \cdot x = x$ for all $a, b \in G$ and $x \in X$. Since all the actions we work with are regular, we drop the adjective regular. The *orbit* of $x \in X$ is denoted by $\mathcal{O}(x) = \{g \cdot x : g \in G\}$.

If $G \times X \to X$ is a regular left action of G on X we consider the induced right action of G on $\Bbbk[X]$, defined as follows. If $f \in \Bbbk[X]$ and $g \in G$, then $(f \cdot g)(x) = f(gx)$. It is well known that G-stable closed subset of X correspond to G-stable radical ideals of $\Bbbk[X]$. We say that $f \in \Bbbk[X]$ is *G-invariant* if $f \cdot g = f$ for any $g \in G$. The set of G-invariants ${}^G\Bbbk[X]$ forms a $\Bbbk$-subalgebra of $\Bbbk[X]$, possibly non-finitely generated. Analogous considerations can be made if we start with a right action $X \times G \to X$.

A *finite dimensional (rational) G-module* is a finite dimension $\Bbbk$-vector space V together with a left action of G on V by linear automorphisms. A right action of G on V defines, in a similar way, the notion of a *right G-module*.

Recall that an *algebraic monoid* M is an algebraic variety together with an associative product $m : M \times M \to M$ with neutral element 1, such that m is a morphism of algebraic varieties. We denote the *set of idempotent elements* of M by $E(M) = \{e \in M : e^2 = e\}$. We denote the *unit group* of M by $G(M) = \{g \in M : \exists\, g^{-1} \in M\,,\ g^{-1}g = gg^{-1} = 1\}$. It is known that $G(M)$ is an algebraic group, open in M (see [9] and [11]).

2.1. Characters of affine algebraic monoids

In this section we establish the basic facts about *extendible characters* for the case of linear algebraic monoids.

Definition2.1. Let M be an algebraic monoid. A *character* of M is a morphism of algebraic monoids $M \to \Bbbk$. We denote the monoid of characters of M by

$$\mathcal{X}(M) = \{\chi \in \Bbbk[M] : \chi(ab) = \chi(a)\chi(b)\ \forall\, a, b \in M\,,\ \chi(1) = 1\}.$$

If $G = G(M)$, then restriction induces an injective morphism of (abstract) monoids $\mathcal{X}(M) \hookrightarrow \mathcal{X}(G)$.

If M is an irreducible affine algebraic monoid, then there exists $n \geq 0$ and a morphism of algebraic monoids $\rho : M \hookrightarrow \mathrm{M}_n(\Bbbk)$, such that ρ is closed immersion (see for example [9, Theorem 3.8]). This motivates the following definition.

Definition2.2. Let M be an irreducible affine algebraic monoid. A character $\chi \in \mathcal{X}(M)$ is called a *determinant* if $\chi^{-1}(0) = M \setminus G(M)$.

By the considerations above, determinants always exists.

Remark2.3. (1) Observe that if $\det M \to \Bbbk$ is a determinant, then $G(M) = M_{\det}$. In particular $\Bbbk\big[G(M)\big] = \Bbbk[M]_{\det}$.
(2) Clearly, $\mathcal{X}(M) \subset \mathcal{X}\big(G(M)\big)$ is a unital submonoid that generates $\mathcal{X}\big(G(M)\big)$ as a group. Indeed, let $\det : M \to \Bbbk$ be a determinant and $\chi \in \mathcal{X}\big(G(M)\big)$. Then there exists $f \in \Bbbk[M]$ and $n \geq 0$ such that $\chi = \frac{f}{\det^n}$. It follows that $f|_G$ is a character, and hence, by continuity, $f \in \mathcal{X}(M)$.

The notion of an *extendible character* is very useful in the study of observable subgroups of algebraic groups. We extend this notion to the setting of algebraic monoids.

Definition2.4. Let M be an affine algebraic monoid and let $H \subset G(M)$ be a closed subgroup. A non-trivial character $\chi \in \mathcal{X}(H)$ is *extendible* if there exists a non-zero *semiinvariant element of weight* χ; that is, there exists $f \in k[M]$ such that $f \cdot x = \chi(x)f$ for all $x \in H$. Such an element is called an *extension of* χ. We denote the monoid of extendible characters of H by $E_M(H)$.

Clearly, restriction to $G = G(M)$ induces an injective homomorphisms of (abstract) monoids $E_M(H) \hookrightarrow E_G(H)$.

Remark2.5. Observe that if χ is an extendible character and f is an extension of χ, we can suppose that $f(1) = 1$. Then $f(x) = (f \cdot x) = (1)\chi(x)f(1) = \chi(x)$ for all $x \in N$, and thus f is an extension of χ to a regular function of M.

Definition2.6. Let M be an affine algebraic monoid and V a finite rational M-module. For every $\alpha \in V^*$ and $v \in V$ we define $\alpha|v : M \to \Bbbk$ as $(\alpha|v)(x) = \alpha(x \cdot v)$ for all $x \in M$. A function of this form is called an *V–representative function* or simply a *representative function.*

Definition2.7. Let G be an algebraic group and let $\chi \in \mathcal{X}(G)$ be a character. Let V be a right G-module, with action $\varphi : V \times G \to V$, $\varphi(v, g) = v \cdot g$. Then the *twisted representation* V_χ is defined as follows. As a vector space $V = V_\chi$, and the action $V_\chi \times G \to V_\chi$ is given by $v \star g = \chi(g)(v \cdot g)$, for every $g \in G$, $v \in V_\chi$.

The following theorem is a straightforward generalization of the corresponding (well known) result for algebraic groups (see [4, Theorem 7.2.3]). We include a proof for the sake of completeness.

Theorem 2.8. *Let M be an affine algebraic monoid with unit group G, $H \subset G$ a closed subgroup and V a finite dimensional rational right H-module. There exists a finite dimensional rational right M-module W, an extendible character $\chi : H \to G_m$ and an injective morphism $\iota : V \to (W|_H)_{\chi^{-1}}$.*

Proof. Given V as above, we proceed as in the proof of [4, Theorem 7.2.3]. It is well known that there exists an injective morphism of H-modules $\theta : V \to \bigoplus_I \Bbbk[H]$, where I is a finite set of indexes (see for example [4, Theorem 4.3.13]). Consider the H-morphism $\alpha = \bigoplus \pi : \bigoplus_I \Bbbk[M] \to \bigoplus_I \Bbbk[H]$, where $\pi : \Bbbk[M] \to \Bbbk[H]$ is the canonical projection.

Call $V' = \alpha^{-1}\big(\theta(V)\big) \subset \bigoplus_I \Bbbk[M]$ and β the restriction of α to the H–submodule V':

$$\begin{tikzcd} V' \arrow[r, hook] \arrow[d, "\beta"'] & \bigoplus_I \Bbbk[M] \arrow[d, "\alpha"] \\ V \arrow[r, "\theta"'] & \bigoplus_I \Bbbk[H] \end{tikzcd}$$

Let $\mathcal{F}$ be a finite $\Bbbk$-linear basis of V and let $\mathcal{F}_0 \subset V'$ be a finite set such that $\beta(\mathcal{F}_0) = \mathcal{F}$. Call R the finite dimensional G-submodule of $\bigoplus_I \Bbbk[M]$ generated by $\mathcal{F}_0$; then R is an M-module. Let S be finite dimensional H-submodule of $R|_H$ (contained in V') generated by $\mathcal{F}_0$. In this way, we produce an exact sequence of rational H–modules $0 \to U \to S \to V \to 0$. Call $n = \dim_{\Bbbk} U$ and consider the commutative diagram below

$$\begin{tikzcd} 0 \arrow[r] & U \otimes \bigwedge^n U \arrow[r] \arrow[d] & S \otimes \bigwedge^n U \arrow[r] \arrow[d] & V \otimes \bigwedge^n U \arrow[r] \arrow[dl, dotted, "\varphi"] & 0 \\ 0 \arrow[r] & U \wedge \bigwedge^n U = 0 \arrow[r] & S \wedge \bigwedge^n U & & \end{tikzcd}$$

where all the solid arrows are the canonical ones, the first row is exact and in the second row the term $U \wedge \bigwedge^n U$ equals zero by dimensional reasons — notice that all the exterior products are taken inside the exterior algebra $\bigwedge R$. One can now prove that φ is bijective (see the proof of [4, Theorem 7.2.3]).

As S and U are H-submodules of $R|_H$, we can view φ as an injective H–morphism $\varphi_1 : V \otimes \bigwedge^n U \to \bigwedge^{n+1} R|_H$. Call χ the rational character associated to the one dimensional H-module $\bigwedge^n U$, i.e. the character defined by the formula $x \cdot u = \chi(x)u$ for all $x \in H$. The map $\varphi_2 : V \to \bigwedge^{n+1} R$,

$\varphi_2(m) = \varphi_1(m \otimes u)$, satisfies that for all $x \in H$,

$$\begin{aligned}\varphi_2(m \cdot x) &= \varphi_1\big((m \cdot x) \otimes u\big) = \varphi_1\big(\chi^{-1}(x)(m \cdot x \otimes u \cdot x)\big)\\ &= \chi^{-1}(x)\varphi_1\big((m \otimes u) \cdot x\big) = \chi^{-1}(x)\varphi_1(m \otimes u) \cdot x\\ &= \chi^{-1}(x)\varphi_2(m) \cdot x.\end{aligned}$$

Hence, if we let $W = \bigwedge^{n+1} R$ and $\iota = \varphi_2$, the proof of the theorem will be complete once we prove that the character χ is extendible. Consider the M–module $\bigwedge^n R$ and let $\alpha \in (\bigwedge^n V)^*$ be such that $\alpha(u) = 1$. Then the representative function $\alpha|u \in \Bbbk[M]$ is an extension of χ. Indeed, if $x \in H$ and $m \in M$, then

$$\big((\alpha|u) \cdot x\big)(m) = (\alpha|u)(xm) = \alpha\big(u \cdot (xm)\big) = \chi(x)\alpha(u \cdot m) = \chi(x)(\alpha|u)(m)$$

and $(\alpha|u)(1) = \alpha(u) = 1$. So that $\alpha|u$ is an extension of χ. $\square$

Let M be an affine algebraic monoid with unit group G. Many results about extendible characters for subgroups of G apply to the situation of closed subgroups of M.

Lemma 2.9. *Let M be an affine algebraic monoid with unit group G, and $H \subset G$ a subgroup, closed in M.*

(1) Let $\rho \in E_M(H)$ be an extendible character and let $f \in k[M]$ be a semiinvariant of weight ρ. Consider the left action $M \times \Bbbk[M] \to \Bbbk[M]$ given by $(x \cdot f)(m) = f(mx)$, $x, m \in M$. Then for any $x \in M$, $x \cdot f$ is also a semiinvariant of weight ρ.

(2) If $\pi : \Bbbk[M] \to \Bbbk[H]$ is the canonical projection and $\rho \in E_M(H)$ is an extendible character, there exists an extension $f \in k[M]$ of ρ such that $\pi(f) = \rho$.

(3) A character $\rho \in \chi(H)$ is extendible if and only if there exists a rational right M–module V and an injective morphism of right H–modules $\iota : \Bbbk\rho \to V$. In other words ρ is extendible if and only if there exists a rational right M–module V and a non-zero element $v \in V$ such that $v \cdot x = \rho(x)v$ for all $v \in H$. Moreover, the right M–module V can be taken to be a finite dimensional right M–submodule of $\Bbbk[M]$.

(4) For any $\gamma \in \mathcal{X}(H)$ there exists $\rho \in E_M(H)$ such that $\gamma\rho \in E_M(H)$.

Proof. Many of these results are straightforward generalizations of the corresponding results for algebraic groups. See [4, Lemma 7.2.8]. For the sake of completeness we include the proofs of (3) and (4).

To prove (3) suppose that ρ is extendible, let $f \in \Bbbk[M]$ be an extension and call W the rational right G–submodule of $\Bbbk[M]$ generated by f. Then the map $\iota : \Bbbk\rho \to W$, $\iota(\rho) = f$, does the job. Conversely, if one has an injective morphism $\iota : \Bbbk\rho \to W|_{_H}$ and call $n = \iota(\rho)$, then $n \cdot x = \iota(\rho) \cdot x = \iota(\rho \cdot x) = \rho(x)\iota(\rho) = \rho(x)n$. Take now $\alpha \in N^*$ such that $\alpha(n) = 1$ and consider $f = \alpha|n$. Then, $f \cdot x(m) = (\alpha|n)(xm)\alpha\big(n \cdot (xm)\big) = \rho(x)(\alpha|n)(m) = \rho(x)f$ for all $x \in H$, and $f(1) = (\alpha|n)(1) = \alpha(n) = 1$.

To prove (4) consider $V = \Bbbk\gamma$. By Theorem 2.8 there exists an extendible character ρ, a right M–module W and an injective H–morphism $\iota : \Bbbk\gamma \to \big(W|_{_H}\big)_{\rho^{-1}}$. Call $n = \iota(\gamma)$, then for any $x \in H$ we have that $n \cdot x = \iota(\gamma) \cdot x = \rho(x)\iota(\gamma \cdot x) = \rho(x)\gamma(x)\iota(\gamma) = \rho(x)\gamma(x)n$. From (3) we conclude that $\rho\gamma \in E_M(H)$. $\square$

2.2. Observable actions of algebraic groups

Observable subgroups were introduced by Bialynicki-Birula, Hochschild and Mostow in [1]. Since then they have been researched extensively, notably by F. Grosshans. See [5] for a survey on this topic. This basic notion has recently been generalized and reformulated in the context of geometric invariant theory by the authors. See [10].

Definition2.10. Let G be an affine algebraic group and let $H \subset G$ be a closed subgroup. The subgroup H is *observable* in G if and only if for any nonzero H-stable ideal $I \subset \Bbbk[G]$ we have that $I^G \neq (0)$, where we consider the right action of H on $\Bbbk[G]$ given by $(f \cdot h)(a) = f(ha)$, for all $f \in \Bbbk[G]$ $h \in H$ and $a \in G$.

Analogously, one can define the notion of a *right observable subgroup*, by considering the action $H \times \Bbbk[G] \to \Bbbk[G]$, $(h \cdot f)(a) = f(ah)$ for $f \in \Bbbk[G]$, $a \in G$, $h \in H$.

Example2.11. (1) If $U \subset G$ is a closed unipotent subgroup, then U is observable, since any U-module has non-zero invariant elements.

(2) If $H \subset G$ is a normal closed subgroup, then H is observable. This follows from condition (2) of Theorem 2.12 below.

(3) Let $H \subset G$ be a closed subgroup, such that $\mathcal{X}(H) = 1$. Then H is observable in G. This follows for example from condition (6) of Theorem 2.12 below.

We now present a collection of equivalent definitions of observability. The proofs can be found in [4, Thms. 10.2.9 and 10.5.5]. We give a different

proof for the fact that H is observable in G if and only if $E_G(H)$ is a group (equivalence (1) $\iff$ (7) of Theorem 2.12). This will provide some insight into the more general situation of algebraic monoids given in Theorem 3.4.

Theorem 2.12. *Let G be an affine algebraic group and $H \subset G$ a closed subgroup. Then the following conditions are equivalent:*

1. *The subgroup H is observable in G.*
2. *The homogeneous space G/H is a quasi–affine variety.*
3. *For an arbitrary proper and closed subset $C \subsetneq G/H$, there exists an non-zero invariant regular function $f \in \Bbbk[G]^H$ such that $f(C) = 0$.*

Moreover, if G is connected the above conditions are equivalent to any of the following.

(4) $H = \{x \in G : x \cdot f = f,\ \forall f \in \Bbbk[G]^H\}$.
(5) $\big[\Bbbk[G]\big]^H = \big[\Bbbk[G]^H\big]$.
(6) $E_G(H) = \mathcal{X}(H)$.
(7) $E_G(H)$ *is a group.*

Proof. To prove that (1) $\Longrightarrow$ (7), let H be observable in G and $\chi \in E_G(H)$ an extendible character. Let $f \in \Bbbk[G]$ be a semiinvariant of weight χ. Then the ideal $I = f\Bbbk[G]$ is H-stable, and hence there exists $g \in \Bbbk[G]$ such that $(fg) \cdot x = fg$ for all $x \in H$. It follows that $g \cdot x = \chi^{-1}(x)g$ for all $x \in H$. Thus $\chi^{-1} \in E_G(H)$.

To prove that (7) $\Longrightarrow$ (1), let $I \subset \Bbbk[G]$ be a non-zero H-stable ideal, and consider $H_{\text{uni}} = R_u(H)[H,H]$. Then H_{uni} is normal in H, with $\mathcal{X}(H_{\text{uni}}) = \{1\}$, and such that $H_{\text{uni}}\backslash H$ is a torus; in particular $\mathcal{X}(H) = \mathcal{X}(H_{\text{uni}}\backslash H)$, since $\mathcal{X}(H_{\text{uni}}) = \{1\}$. It follows that H_{uni} is observable in G, and thus $I^{H_{\text{uni}}} \neq (0)$ is a H-module. Then $I^{H_{\text{uni}}}$ is a right $(H_{\text{uni}}\backslash H)$-module, and hence

$$(0) \neq I^{H_{\text{uni}}} = \bigoplus_{\chi \in \mathcal{X}(H_{\text{uni}}\backslash H)} I_\chi^{H_{\text{uni}}} = \bigoplus_{\chi \in \mathcal{X}(H)} I_\chi^{H_{\text{uni}}}$$

Hence, there exists $\chi \in \mathcal{X}(H)$ and $f \in I \setminus \{0\}$ such that $f \cdot x = \chi(x)f$ for all $x \in H$. Thus $\chi \in E_G(H)$. Since $E_G(H)$ is a group, then $\chi^{-1} \in E_G(H)$ and hence there exists $g \in \Bbbk[G]$ such that $g \cdot x = \chi^{-1}(x)g$ for all $x \in H$. It follows that $0 \neq fg \in I^H$. $\square$

Definition2.13. Let G be an affine algebraic group acting on an affine variety X. We say that the action is *observable* if for any nonzero G-stable ideal $I \subset \Bbbk[X]$ we have that $I^G \neq (0)$.

Example2.14. (1) The action of an unipotent group on an affine variety is always observable, since for every right U-module M, we have that $M^U \neq 0$.
(2) Let G be an algebraic group and H a closed subgroup. Then the action of H on G by left translations is observable if and only if H is observable in G, in the sense of Definition 2.10, if and only if the action by right translations is observable.

Observable actions have been studied in detail in [10], where several characterizations of this important property were presented. We collect here the ones that we need in what follows. We begin by recalling some notation.

Definition2.15. Let G be an affine group acting on an affine variety X. We let

$$\Omega(X) = \big\{ x \in X : \dim \mathcal{O}(x) \text{ is maximal and } \overline{\mathcal{O}(x)} = \mathcal{O}(x) \big\}.$$

That is, $\Omega(X)$ is the set of orbits of maximal dimension that are closed. The reader should be aware that $\Omega(X)$ can be empty.

Theorem 2.16. *Let G be a connected affine algebraic group acting on an irreducible affine variety X. Then the following are equivalent*

1. *The action is observable.*
2. *(i)* $\big[\Bbbk[X]^G\big] = \big[\Bbbk[X]^G\big]$ *and*
   (ii) $\Omega(X)$ *contains a non-empty open subset.*
3. *There exists a nonzero invariant $f \in \Bbbk[X]^G$ such that the action of G on X_f is observable.*

Proof. See [10, Proposition 3.2 and Theorem 3.10]. □

If G is a reductive group acting on an affine variety X, then a result of Popov (see [6, Theorem 4]) guarantees that if $\Omega(X)$ is nonempty then it is an open set. In [10] this result was used to prove that in this case, condition (2ii) of Theorem 2.16 is sufficient to guarantee observability.

Proposition 2.17. *Let G be a reductive group acting on an affine variety X. Then the action is observable if and only if $\Omega(X) \neq \emptyset$.*

Proof. See [10, Theorem 4.7]. □

2.3. The Affinized quotient

Let G be an affine algebraic group acting on the affine variety X. It is well known that the categorical quotient does not necessarily exist, even when $\Bbbk[X]^G$ is finitely generated. However, if $\Bbbk[X]^G$ is finitely generated, then $\operatorname{Spec}(\Bbbk[X]^G)$ satisfies a universal property in the category on the affine algebraic varieties.

Definition2.18. Let G be an affine algebraic group acting on an affine variety X, in such a way that $\Bbbk[X]^G$ is finitely generated. The *affinized quotient* of the action is the morphism $\pi : X \to X/_{\mathrm{aff}}\, G = \operatorname{Spec}(\Bbbk[X]^G)$.

It is clear that π satisfies the following universal property.

Let Z be an affine variety and $f : X \to Z$ a morphism constant on the G-orbits. Then there exists a unique $\widetilde{f} : X/_{\mathrm{aff}}\, G \to Z$ such that the following diagram is commutative.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Z \\ X/_{\mathrm{aff}}\, G \arrow[ur, "\widetilde{f}"'] & \end{tikzcd}$$

Indeed, it is clear that the induced morphism $f^* : \Bbbk[Z] \to \Bbbk[X]$ factors trough $\Bbbk[X]^G$.

Remark2.19. It is clear that the morphism $\pi : X \to X/_{\mathrm{aff}}\, G$ is dominant. However, π is not necessarily surjective. For example, consider a semisimple group $X = H$ and its maximal unipotent subgroup G acting on X by left translation.

3. Observable subgroups of algebraic monoids

We now adapt the notion of observability to the situation of subgroups of algebraic monoids.

Definition3.1. Let M be an algebraic monoid with unit group G, and let $H \subset G$ be a closed subgroup. We say that H is *(left) observable in M* if the action by left multiplication $H \times M \to M$, $h \cdot m = hm$, is observable in the sense of Definition 2.13.

Similarly, we say that H is *right observable in M* if the action by right multiplication $M \times H \to M$, $m \cdot h = mh$, is observable.

Example3.2. (1) Since the action of a unipotent group on an affine variety is observable (all orbits are closed), it follows that any unipotent subgroup U of $G(M)$ is observable in M.

(2) If $M = G(M)$ is an algebraic group, then Definitions 2.10 and 3.1 coincide.

(3) If follows from Proposition 2.17 that if $H \subset G(M)$ is reductive, and closed in M, then H is observable in M. Indeed, if $H = \overline{H}$, then $\Omega(M) \neq \emptyset$.

Theorem 3.3. *Let M be an affine irreducible algebraic monoid with unit group G, and let $H \subset M$ be an observable subgroup. Then*

(1) The subgroup H is observable in G.

(2) H closed in M.

(3) $\big[\Bbbk[M]\big]^H = \big[\Bbbk[G]\big]^H = \big[\Bbbk[G]^H\big] = \big[\Bbbk[M]^H\big]$.

(4) The subgroup H satisfies

$$\begin{aligned} H = \big\{x \in G : f \cdot x = f,\ \forall f \in \Bbbk[G]^H\big\} = \\ \big\{x \in G : f \cdot x = f,\ \forall f \in \Bbbk[M]^H\big\} = \\ \big\{x \in M : f \cdot x = f,\ \forall f \in \Bbbk[M]^H\big\}^0. \end{aligned}$$

Recall that if N is an algebraic monoid, then N^0 is the unique irreducible component containing 1*, see for example [12, Thm. 4].*

Proof. (1) Let $C \subset G$ be a H-stable closed subset. Then $\overline{C} \subset M$ is H-stable, and it follows that there exists a $f \in \mathcal{V}(\overline{C})^H \setminus \{0\} \subset \Bbbk[M]$. Since G is open in M, it follows that $f \in \mathcal{V}(C)^H \setminus \{0\} \subset \Bbbk[G]$.

(2) Since H is observable in M, it follows from Theorem 2.16 that $\Omega(M)$ contains a non-empty open subset. Since $G = \bigcup Hg$, G is contained in $M_{\max}$, and hence there exists $g \in G$ such that $g \in \Omega(M) \cap G$, i.e. such that Hg is closed in M. Since multiplication by an element of g is an isomorphism, it follows that H is closed in M.

(3) First observe that $\big[\Bbbk[M]^H\big] \subset \big[\Bbbk[M]\big]^H = \big[\Bbbk[G]\big]^H$. Let $g \in \big[\Bbbk[M]\big]^H$, and consider the ideal $I = \big\{f \in \Bbbk[M] : fg \in \Bbbk[M]\big\}$. Then I is a non-zero H-stable ideal, and hence there exists $h \in I^H \setminus \{0\}$. It follows that $r = hg \in \Bbbk[M]^H$ and hence $g = \frac{r}{h} \in \big[\Bbbk[M]^H\big]$. The remaining equality follows from Theorem 2.12.

(4) The first equality follows from Theorem 2.12. It is clear that $A = \big\{x \in G : f \cdot x = f,\ \forall f \in \Bbbk[G]^H\big\} \subset B = \big\{x \in G : f \cdot x = f,\ \forall f \in \Bbbk[M]^H\big\}$. Let $x \in B$ and $f \in \Bbbk[G]^H$. By (3), it follows that $f \in \big[\Bbbk[G]^H\big] = \big[\Bbbk[M]^H\big]$. Hence, there exist $g, h \in \Bbbk[M]^H$ such that $f = \frac{g}{h}$. It follows that $f \cdot x = \frac{g \cdot x}{h \cdot x} = \frac{g}{h}$.

In order to prove the last equality, we first observe that $N = \mathcal{V}\big(\{x \mapsto (f\cdot x)(m) - f(m)\,,\ m \in M\,,\ f \in \Bbbk[M]^H\}\big)$ is a closed submonoid of M. Since G is open in M, it follows that

$$H = \overline{H} = \overline{\overline{H} \cap G} = N^0.$$

$\square$

Theorem 3.4. *Let M be an irreducible affine algebraic monoid with unit group G and let $H \subset G$ be a subgroup, closed in M. Then the following conditions are equivalent.*

(1) The subgroup H is observable in M.

(2) $E_M(H)$ is a group. That is, for every $\rho \in E_M(H)$, $\rho^{-1} \in E_M(H)$.

(3) $E_M(H) = \mathcal{X}(H)$, i.e. every rational character is extendible.

(4) For every finite dimensional rational right H-module V there exists a finite dimensional rational right M-module W and an injective morphism of H-modules $\xi : V \to W|_{H}$.

(5) H is observable in G and for any (some) determinant $\det : M \to \Bbbk$, *then* $\frac{1}{\det}|_{H} \in E_M(H)$.

Proof. In order to prove that (1) implies (2), assume that H is observable in M, and let $\chi \in E_M(H)$ be an extendible character. Let g be an extension of χ. Then the ideal $g\Bbbk[M]$ is a nonzero H-stable ideal and hence there exists $f \in \Bbbk[M]$ such that $(gf)\cdot x = gf$ for all $x \in H$. It follows that $f\cdot x = \chi^{-1}(x)f$ for all $x \in H$. That is, $\chi^{-1} \in E_M(H)$.

Since $E_M(H)$ generates $\mathcal{X}(H)$ as a group, it is clear that conditions (2) and (3) are equivalent.

To prove that (2) implies (1), assume that $E_M(H)$ is a group and let $I \subset \Bbbk[M]$ be a non-zero H-stable ideal. Let $H_{\mathrm{uni}} = R_u(H)[H,H]$, as in the proof of Theorem 2.12. Then H_{uni} is observable in H. Consider a determinant $\det : M \to \Bbbk$. Since $\mathcal{X}(H_{\mathrm{uni}}) = 1$, it follows that $H_{\mathrm{uni}} \subset \det^{-1}(1)$. Thus, we have found an H_{uni}-invariant regular function $\det \in \Bbbk[M]^{H_{\mathrm{uni}}}$, such that $\Bbbk[G] = \Bbbk[M]_{\det}$. It follows from Theorem 2.16 that H_{uni} is observable in M, and thus the right H-module $I^{H_{\mathrm{uni}}}$ is not trivial. Then, as in the proof of Theorem 2.12, it follows that $I^{H_{\mathrm{uni}}}$ is a right $(H_{\mathrm{uni}}\backslash H)$-module, and hence

$$(0) \neq I^{H_{\mathrm{uni}}} = \bigoplus_{\chi \in \mathcal{X}(H_{\mathrm{uni}}\backslash H)} I_\chi^{H_{\mathrm{uni}}} = \bigoplus_{\chi \in \mathcal{X}(H)} I_\chi^{H_{\mathrm{uni}}}.$$

Hence, there exists $\chi \in \mathcal{X}(H)$ and $f \in I \setminus \{0\}$ such that $f \cdot x = \chi(x)f$ for all $x \in H$. Thus $\chi \in E_M(H)$. Since $E_M(H)$ is a group, it follows that

$\chi^{-1} \in E_M(H)$. Thus, there exists $g \in \Bbbk[G]$ such that $g \cdot x = \chi^{-1}(x)g$ for all $x \in H$. It follows that $0 \neq fg \in I^H$.

To prove that (2) implies (4), we follow the idea of the proof of [4, Theorem 10.2.9]. If V is a finite dimensional rational right H-module, using Theorem 2.8 we deduce the existence of an extendible character ρ, a finite dimensional rational right M-module W and an injective map $\iota : V \to W$ such that $\iota(m \cdot x) = \rho^{-1}(x)\iota(m) \cdot x$ for all $x \in H$. By hypothesis the character ρ^{-1} is extendible, so if we take $f \in \Bbbk[M]$ that extends ρ^{-1} and call W_0 the right M-submodule of $\Bbbk[M]$ generated by f, we can define an injective map $\xi : V \to W \otimes W_0$, given as $\xi(m) = \iota(m) \otimes f$. If we endow $W \otimes W_0$ with the diagonal right M-module structure, the following computation shows that ξ is H-equivariant. Let $x \in H$. Then

$$\begin{aligned}\xi(m \cdot x) &= \iota(m \cdot x) \otimes f = \rho^{-1}(x)(\iota(m) \cdot x) \otimes f = (\iota(m) \cdot x) \otimes \rho^{-1}(x)f \\ &= (\iota(m) \cdot x) \otimes (f \cdot x) = \xi(m) \cdot x.\end{aligned}$$

We now prove that (4) implies (3). Let $\gamma \in \mathcal{X}(H)$ be a rational character of H and consider the rational H-module $V = \Bbbk\gamma$. If ξ and W are as in condition (4) we conclude, using Lemma 2.9, that γ is extendible.

Assume now that (1) holds. Then by Theorem 3.3, H is observable in G, and it follows from (2) that $\frac{1}{\det} \in E_M(H)$. Hence, (5) holds.

Finally, if (5) holds, then by Theorem 2.12, $\mathcal{X}(H) = E_G(H)$. Let $\det \in \mathcal{X}(M)$ be a determinant such that $\frac{1}{\det}|_H \in E_M(H)$, and let $f \in \Bbbk[M]$ be an extension of $\frac{1}{\det}|_H$. If $\chi \in \mathcal{X}(H)$, let $g \in \Bbbk[G]$ be an extension of χ. Then there exists $l \in \Bbbk[M]$ and $n \geq 0$ such that $g = \frac{l}{\det^n}$. Therefore, for every $x \in H$ and $a \in G$, we have that

$$\chi(x)\frac{l(a)}{\det^n(a)} = \chi(x)g(a) = (g \cdot x)(a) = \frac{(l \cdot x)(a)}{\det^n(x)\det^n(a)}.$$

It follows that $l \cdot a = \chi(x)\det^n(x)l$; that is $l \in \Bbbk[M]$ is an extension of $\chi \det^n \in \mathcal{X}(H)$. Then $lf \in \Bbbk[M]$ is an extension of χ. $\square$

Theorem 3.5. *Let M be an irreducible affine algebraic monoid with unit group G and let $H \subset G$ be a subgroup, closed in M. Then the following conditions are equivalent.*

(1) The subgroup H is observable in M.

(2) There exists a finite dimensional rational right M-module V and $v \in V$ such that $G_v = H$.

In particular, if condition (2) holds, then $G_v = M_v$ is closed in M.

Proof. To prove that condition (1) implies condition (2), we adapt the proof of the case $M = G$ (see [4, Corollary 7.3.6]). First we observe that, since G is observable in M, H is observable in G and $\big[\Bbbk[G]^H\big] = \big[\Bbbk[G]\big]^H = \big[\Bbbk[M]\big]^H = \big[\Bbbk[M]^H\big]$. Let $\{u_0, u_1, \dots, u_n\} \subset \Bbbk[M]^H$ be such that $\big\{\frac{u_1}{u_0}, \dots, \frac{u_n}{u_0}\big\}$ generates $\big[\Bbbk[M]\big]^H$ over $\Bbbk$. Let $W \subset \Bbbk[M]$ be the finite dimensional rational right M-submodule generated by $u_0, \dots, u_n$. Let $V = \bigoplus_{i=0}^n W$, $v_0 = (u_0, \dots, u_n) \in V$ and consider the stabilizer G_{v_0}. It is clear that $H \subset G_{v_0}$.

Conversely, if $a \in G_{v_0}$, then $u_i \cdot a = u_i$, $i = 0, \dots, n$, and hence $\frac{u_i}{u_0} \cdot a = \frac{u_i}{u_0}$, $i = 1, \dots, n$. As the elements $\frac{u_i}{u_0}$, $i = 1, \dots, n$, generate $\big[\Bbbk[M]\big]^H$, we conclude that $f \cdot a = f$ for all $f \in \big[\Bbbk[M]\big]^H = \big[\Bbbk[G]\big]^H$. Hence, $a \in H$, as follows by example from [4, Corollary 7.3.4].

To prove that (2) implies (1), let v, V be as in (2) and consider the following commutative diagram

$$\begin{tikzcd} M \arrow[r] & \overline{z \cdot M} \\ G \arrow[u, hook] \arrow[r] & v \cdot G \arrow[u, hook] \end{tikzcd}$$

Since G/H is quasi-affine, it follows that H is observable in G, and hence $\big[\Bbbk[G]^H\big] = \big[\Bbbk[G]\big]^H$. Thus,

$$\big[\Bbbk[\overline{v \cdot M}]\big] = \Bbbk(v \cdot G) \cong \big[\Bbbk[G]^H\big] = \big[\Bbbk[G]\big]^H.$$

On the other hand, the orbit morphism $M \to \overline{v \cdot M}$ is dominant, and hence induces an inclusion $\iota : \Bbbk[\overline{v \cdot M}] \hookrightarrow \Bbbk[M]$. Since $G_v = H$, it follows that $f\big(v \cdot (hm)\big) = f(v \cdot m)$ and hence $\iota\big(\Bbbk[\overline{v \cdot M}]\big) \subset \Bbbk[M]^H$. Thus,

$$\big[\Bbbk[M]\big]^H = \big[\Bbbk[G]\big]^H = \big[\Bbbk[\overline{v \cdot M}]\big] \subset \big[\Bbbk[M]^H\big] \subset \big[\Bbbk[M]\big]^H,$$

and hence the action by right multiplication of H on M is observable.

Finally, recall that observable subgroups are closed in M. $\square$

Remark3.6 (Open question). It is clear that all the results of this section remain valid when considering right observability simply by adapting the statements and proofs in an obvious way.

If $M = G(M)$ is an algebraic group, it is well known that $H \subset G$ is (left) observable in G if and only if H is right observable. Indeed, in this case the antipode $S : \Bbbk[G] \to \Bbbk[G]$, $S(f)(x) = f(x^{-1})$, induces an isomorphism

$\Bbbk[G]^H \cong {}^H\Bbbk[G]$. In the more general case of an algebraic monoid M, this line of reasoning cannot be applied, since $S\big(\Bbbk[M]\big)$ is not included in $\Bbbk[M]$. This raises the following question.

Q1 *Let M be an algebraic monoid with unit group G, and $H \subset G$ a closed subgroup, (left) observable in M. Is H right observable in M?*

4. Quotients of monoids by normal subgroups

Let M be an algebraic monoid of unit group G and let $H \subset G$ a closed subgroup, observable in M. Our goal is to prove that if H is normal in G, then the affinized quotient $M/_{\mathrm{aff}} H$ is an affine monoid with unit group G/H. See Theorem 4.4. Before doing so, we present a general result about the affinized quotient of a monoid by an observable subgroup.

Proposition 4.1. *Let M be an algebraic monoid with unit group G and let $H \subset G$ be a closed subgroup, observable in M. Assume that $\Bbbk[M]^H$ is finitely generated. Then $M/_{\mathrm{aff}} H$ is an affine embedding of G/H. That is, $M/_{\mathrm{aff}} H$ is an affine G-variety, with an open orbit isomorphic to G/H.*

Proof. Consider the following commutative diagram

$$\begin{tikzcd} G \arrow[r, hook] \arrow[d, two heads] & M \arrow[d, "\pi"] \\ G/H \arrow[r, "\varphi"'] & M/_{\mathrm{aff}} H \end{tikzcd}$$

where the existence of φ follows from the universal property of the quotient. Since H is observable in M, it follows from Theorem 3.3 that $\Bbbk(M/_{\mathrm{aff}} H) = \big[\Bbbk[M]^H\big] = \big[\Bbbk[M]\big]^H = \big[\Bbbk[G]\big]^H = \big[\Bbbk[G]^H\big]$. Hence, φ is a birational dominant G-morphism. Since G/H is a homogeneous G-space, it follows that φ is an open immersion. $\square$

Proposition 4.2. *Let M be a reductive monoid with unit group G, and let $H \subset G$ be a normal subgroup, closed in M. Then the categorical quotient $M//H$ is an affine algebraic monoid with unit group G/H. Furthermore, $\pi : M \to M//H$ is a morphism of algebraic monoids.*

Proof. First of all, we prove that H is observable in M. Indeed, H is a normal subgroup of G and thus it is a reductive group. Since $H \subset M$ is

closed, it follows that gH is closed in M for all $g \in G$, and we deduce from Proposition 2.17 that H is observable in M.

Let $(G\times G)\times \Bbbk[M] \to \Bbbk[M]$, $\big((a,b)\cdot f\big)(m) = f(a^{-1}mb)$ $a,b \in G$, $m \in M$, be the canonical action. If $f \in \Bbbk[M]^H$ and $c \in H$, then

$$\begin{aligned}\big(c\cdot\big((a,b)\cdot f\big)\big)(m) = \big((c,1)\cdot\big((a,b)\cdot f\big)\big)(m) = f(a^{-1}c^{-1}mb) = \\ f(la^{-1}mb) = f(a^{-1}mb) = \big((a,b)\cdot f\big)(m)\end{aligned}$$

where $l \in H$ is such that $a^{-1}c^{-1} = la^{-1}$. If follows that $\Bbbk[M]^H$ is a $(G\times G)$-submodule of $\Bbbk[M]$. Let now $(c,d) \in H\times H$. Then $(1,d)\cdot f = f$. Indeed, if $g \in G$, then $gd = lg$ for some $l \in H$, and hence $f(gd) = f(lg) = f(g)$. In other words, $(1,d)\cdot f|_G = f|_G$, and thus $(1,d)\cdot f = f$. It follows that $(c,d)\cdot f = f$. Hence, $M//H$ is an affine $(G/H\times G/H)$-variety and the coproduct $m^* : \Bbbk[M] \to \Bbbk[M]\otimes\Bbbk[M]$ is such that

$$m^*\big(\Bbbk[M]^H\big) \subset \big(\Bbbk[M]^H \otimes \Bbbk[G]\big) \cap \big(\Bbbk[G]\otimes\Bbbk[M]^H\big) = \Bbbk[M]^H \otimes \Bbbk[M]^H.$$

Moreover, since $\Bbbk[M]^H \subset \Bbbk[G]^H = \Bbbk[G/H]$, it follows that $M//H$ is an algebraic monoid and that we have a commutative diagram

$$\begin{tikzcd} G \arrow[r, hook] \arrow[d] & M \arrow[d, "\pi"] \\ G/H \arrow[r, "\varphi"'] & M//H \end{tikzcd}$$

where the existence of φ follows from the universal property of the quotient. Then φ is a dominant morphism of algebraic monoids, and hence $G(M//H) = \varphi(G/H)$. Since H is observable in M it follows that

$$\Bbbk(G/H) = \big[\Bbbk[G]\big]^H = \big[\Bbbk[M]\big]^H = \big[\Bbbk[M]^H\big] = \Bbbk(M//H).$$

Thus, $\varphi : G/H \to M//H$ is an injective birational G-morphism, and it follows that $\varphi : G/H \to G(M//H)$ is an open immersion. Hence, φ is an isomorphism. □

The following result (with the exception of the last assertion) follows from [8, Theorem 2.5] and [9, Theorem 6.1].

Theorem 4.3. *Let M be an affine algebraic monoid with unit group G. Then $\Bbbk[M]^{R_u(G)}$ is a finitely generated algebra. Moreover, $M/_{\mathrm{aff}}\, R_u(G) = \mathrm{Spec}\big(\Bbbk[M]^{R_u(G)}\big)$ is an affine algebraic monoid with unit group $G/R_u(G)$. The morphism $\pi : M \to M/_{\mathrm{aff}}\, R_u(G)$ is a surjective morphism of algebraic*

monoids, and satisfies the following universal property. For any morphism $f : M \to N$, of algebraic monoids, such that $f\big(R_u(G)\big) = \{1_N\}$ there exists a unique morphism $\widetilde{f} : M/_{\mathrm{aff}}\, R_u(G) \to N$ such that the diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\pi"'] & N \\ M/_{\mathrm{aff}}\, R_u(G) \arrow[ur, "\widetilde{f}"'] & \end{tikzcd}$$

commutes. Moreover, the subgroup $H \subset G$ is closed in M if and only if $\pi(H) = \big(HR_u(G)\big)/R_u(G) \subset G/R_u(G)$ is closed in $M/_{\mathrm{aff}}\, R_u(G)$.

Proof. We provide a proof of the last assertion, since it is not proved in [8] or [9].

It follows from [7, Theorem 3.18] that $H = \overline{H}$ if and only if $E(\overline{H}) = \{1\}$. But by [7, Corollary 6.10], $E(\overline{H}) = \bigcup_{a \in H} aE(\overline{S})a^{-1}$, where $S \subset H$ is a maximal tours of H. Hence, $H = \overline{H} \subset M$ if and only if $S = \overline{S} \subset M$ for any (some) maximal torus $S \subset H$.

By [8, Theorem 2.5], it follows that, if $T \subset G$ is a maximal torus of G, then $\pi|_{\overline{T}} : \overline{T} \to \overline{\pi(T)} \subset M/_{\mathrm{aff}}\, R_u(G)$ is an isomorphism. Let now $S \subset H$ be a maximal torus of H and $T \subset G$ a maximal torus of G such that $S \subset T$. Then $\pi|_{\overline{S}} : \overline{S} \to \overline{\pi(S)} \subset \overline{\pi(T)}$ is an isomorphism. In particular, $S = \overline{S} \subset M$ if and only if $\pi(S) = \overline{\pi(S)} \subset M/_{\mathrm{aff}}\, R_u(G)$. Since $\pi(S) \subset \pi(H)$ is a maximal tours, it follows that $H = \overline{H} \subset M$ if and only if $\pi(H) = \overline{\pi(H)} \subset M/_{\mathrm{aff}}\, R_u(G)$.

We conclude the proof by showing that the affinized quotient $\pi : M \to M/_{\mathrm{aff}}\, R_u(G)$ is a surjective morphism of algebraic monoids (this fact is implicit in [8, 9] but was not stated or proved). By construction, π is a morphism of algebraic monoids. Since $M/_{\mathrm{aff}}\, R_u(G)$ is a reductive monoid, it follows that

$$M/_{\mathrm{aff}}\, R_u(G) = G/R_u(G)E\big(M/_{\mathrm{aff}}\, R_u(G)\big)G/R_u(G),$$

see for example [9, Theorem 4.2]. The surjectivity of π follows now from the fact that $\pi\big(E(M)\big) = E\big(M/_{\mathrm{aff}}\, R_u(G)\big)$. $\square$

Theorem 4.4. *Let M be an algebraic monoid with unit group G, and let $H \subset G$ be a normal subgroup, closed in M. Then the action $H \times M \to M$ is observable. Moreover, if $\Bbbk[M]^H$ is finitely generated, then the affinized quotient $M/_{\mathrm{aff}}\, H$ is an affine algebraic monoid, with unit group G/H.*

Proof. Let $\pi : M \to M/_{\mathrm{aff}}\, R_u(G)$ be the affinized quotient as in Theorem 4.3. By [ibid] $M/_{\mathrm{aff}}\, R_u(G)$ is a reductive algebraic monoid with unit group

$G/R_u(G)$. We use π to help us find a determinant function on M (the function μ below) with suitable properties. Consider the normal subgroup $\pi(H) = \pi\big(HR_u(G)\big) \subset G/R_u(G)$. Then $\pi(H)$ is closed in $M/_{\text{aff}}\, R_u(G)$ and hence by Proposition 4.2 it follows that $N = \big(M/_{\text{aff}}\, R_u(G)\big)/\!/\pi(H)$ is an affine algebraic monoid. Let $\rho : M/_{\text{aff}}\, R_u(G) \to N$ be the affinized quotient and $\chi : N \to \Bbbk$ a character such that $\chi^{-1}(0) = N \setminus G(N) = \big(G/R_u(G)\big)/\pi(H) \cong G/HR_u(G)$. Then $\mu : \chi \circ \rho \circ \pi : M \to \Bbbk$ is a character such that $\mu(H) = 1$ and $\mu^{-1}(0) = M \setminus G$. In particular $\mu \in \Bbbk[M]^H$. Since H is normal in G H is observable in G. In other words, the action of H on $G = M_\mu$ is observable. It follows from Theorem 2.16 that the action of H on M is observable.

To prove the last assertion we use the same arguments as in the proof of Proposition 4.2. Since $m^*\big(\Bbbk[M]^H\big) \subset \Bbbk[M]^H \otimes \Bbbk[M]^H$, it follows that $M/_{\text{aff}}\, H$ is an algebraic monoid and that we have a commutative diagram

$$\begin{tikzcd} G \arrow[r, hook] \arrow[d] & M \arrow[d, "\pi"] \\ G/H \arrow[r, "\varphi"'] & M/_{\text{aff}}\, H \end{tikzcd}$$

where the existence of φ follows from the universal property of the quotient. Then φ is a dominant morphism of algebraic monoids, and hence $G(M/_{\text{aff}}\, H) = \varphi(G/H)$. Since the action of H on M is observable, it follows from Theorem 3.3 that $\big[\Bbbk[M]^H\big] = \big[\Bbbk[M]\big]^H$, and hence φ is a birational morphism. Since ρ is G-equivariant, to follows that φ is an open immersion, and thus $G/H \cong G(M/_{\text{aff}}\, H)$. $\square$

If H is normal in $G(M)$ then the affinized quotient $M/_{\text{aff}}\, H$ satisfies a universal property in the category of algebraic monoids.

Proposition 4.5. *Let N be an algebraic monoid and $f : M \to N$ a morphism of algebraic monoids such that $f(H) = \{1_N\}$. Then there exists a morphism of algebraic monoids $\widetilde{f} : M/_{\text{aff}}\, H \to N$ such that the following diagram commutes.*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\pi"'] & N \\ M/_{\text{aff}}\, H \arrow[ur, "\widetilde{f}"'] & \end{tikzcd}$$

Proof. Let

$$1 \longrightarrow L \longrightarrow N \xrightarrow{\alpha} A(N) \longrightarrow 0$$

be the Chevalley decomposition of N, where α is the Albanese morphism (see [2, Theorem 3.2.1] and [3, Theorem 1.1]). Since M is affine, if follows that form the property of the Albanese morphism that $f(M) \subset L$ (see for example [3, Theorem 5.1]). Since f is a morphism of algebraic monoids, such that $f(H) = \{1\}$, it follows that f is constant on the H-orbits, and hence $f^* : \Bbbk[L] \to \Bbbk[M]$ factors through $\Bbbk[M]^H$. In other words, there exists $\widetilde{f} : M/_{\text{aff}} H \to L$ such that $f = \widetilde{f} \circ \pi$. $\square$

Example4.6. Let

$$H = \left\{ \left(\begin{smallmatrix} a^2 & 0 \\ 0 & a^{-1} \end{smallmatrix} \right) : a \in \Bbbk^* \right\} \subset \mathrm{GL}_2(\Bbbk) \subset \mathrm{M}_{2\times 2}(\Bbbk) = M.$$

Then H is observable in $\mathrm{GL}_2(\Bbbk) = G(M)$ and closed in M, but there exists no character $\chi : M \to \Bbbk$ such that $H \subset \chi^{-1}(1)$ and $\chi^{-1}(0) = M \setminus G(M)$. Since H is closed in G and since $\ell_g : M \to M$, $\ell_g(m) = g \cdot m$, is an isomorphism, it follows that $gH \subset M$ is closed in M for all $g \in \mathrm{GL}_2(\Bbbk)$. Since H is reductive, it follows from Proposition 2.17 that the action of H on M is observable.

Remark4.7 (Open question). Example 4.6 shows that the condition of $H \subset G$ being a normal subgroup is crucial in Theorem 4.4. However, in that example the action of H on M is observable. This raises the following question.

Q2 *Let M be an algebraic monoid with unit group G, and let $H \subset G$ an observable group, closed in M. Is the action $H \times M \to M$ observable?*

Remark4.8. If Q2 has a positive answer, then Q1 (see Remark 3.6) has a positive answer. Indeed, assume that Q2 has a positive answer and let $H \subset M$ be a left observable subgroup. Then by Theorem 3.3 it follows that H is observable in G, and hence H is right observable in G. If then follows from Q2 that H is right observable in M.

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